Parahoric subgroups and automorphic forms(WORKSHOP ON ALGEBRAIC GROUPS AND RELATED TOPICS)

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数理解析研究所講究録
第 737 巻 1990 年 94-105

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1 introduction

After Langlands and others, the theory of classical holomorphic automorphic forms is, in a sense, a part of the representation theory of algebraic groups. Of course, the representation theory is more essential and more general than the classical theory. But, some part of the theory can be understood in classical language fairly completely. The theory of new forms of Atkin-Lehner and others belonging to $\Gamma_0(p)$ is one such example.
Here, we treat one such theory, and give some general problems or conjecture on comparison of automorphic forms belonging to two different algebraic groups and also give results in symplectic cases. This is a try to generalize the classical theorem of Eichler and Shimizu, and some approach to a part of the Langlands conjecture.

2 Problems or Conjectures

Let G be a connected quasi-split reductive algebraic group over Q whose semi-simple part is simply connected, and G' be an inner twist of G. We also assume that the symmetric domain attached to G(R) is a bounded symmetric domain D, and that $G'(R)/center$ is either compact or attached to bounded symmetric domain. We would like to compare the classical automorphic forms on these two groups. We want to treat only those forms belonging to minimal parahoric subgroup of each group. Now, we explain this.

We fix a finite subset V of all places of Q, and assume that, for any place v of Q, we have $G_v \cong G'_v$ if and only if $v \notin V$, where $G_v = G(Q_v)$. For each $v \notin V$, we fix some "standard " open subgroup U_v of $G_v \cong G'_v$. (We do not specify which one we should take as U_v at this moment.It need not be maximal.) For each $v \in V$, we fix a minimal parahoric subgroup B_v of G_v. Let $S_{v,aff}$ be the set of generators of affine Weyl group of G_v. Then, the set of all subgroups of G_v which contains B_v corresponds bijectively to the set of all finite subsets of $S_{v,aff}$. For each $\theta \subset S_{v,aff}$, we denote by U_θ the subgroup determined by this bijection. In the same way, for G'_v , we define $S'_{v,aff}$, B'_v, and $U'_{\theta'}$. (When G'_v is compact, we just take G'_v as B'.) Now, put $S = \cup_{v \in V} S_{v,aff}$ and $S' = \cup_{v \in V} S'_{v,aff}$. For each set $\Theta \subsetneq S$, define an open subgroup $\mathcal{U}_\Theta$ of the adelization G_A of G by:

$$\mathcal{U}_\Theta = G_\infty \times \prod_{v \in V} U_{\Theta \cap S_{v,aff}} \times \prod_{v \notin V} U_v$$

In the same way, we define subgroups $\mathcal{U}'_{\Theta'}$ of G'_A for each $\Theta' \subsetneqq S'$. Here, Θ or Θ' might be the empty set.

Since we shall treat everything classically, we review the definition of automorphic forms. For the sake of simplicity, we assume from now on that the semisimple part of G (and also of G' when $G'(R)/center$ is not compact) satisfies the strong approximation theorem. (This assumption is , in a sense, superfluous, but the definition of automorphic forms becomes slightly more complicated without this assumption.) We fix a representation χ of the connected component of the maximal compact subgroup of the semi-simple part of $G(R)$. Denote by $G^0(R)$ the connected component of the semisimple part of $G(R)$ (as the real Lie group), and denote by $J(g, Z)$ ($g \in D$, $Z \in D$) the canonical automorphic factor attached to $G^0(R)$. We put $\Gamma_\Theta = (G(Q) \cap \mathcal{U}_\Theta) \cap G^0(R)$. Then the space $S_\chi(\mathcal{U}_\Theta)$ of cusp forms belonging to $\mathcal{U}_\Theta$ is defined to be the set of those holomorphic functions f on D such that

$$f(\gamma(Z)) = \chi(J(\gamma, Z))f(Z) \text{ for all } \gamma \in \Gamma_\Theta$$

and that f vanishes on each boundary of the Satake-Baily-Borel compactification. When $G'(R)$ is compact modulo center, then we take a representation $(\chi', V_{\chi'})$ of $G'(R)$ and the space $S_{\chi'}(\mathcal{U}'_\Theta)$ of automorphic forms on G'_A is defined as usual (cf. [1]) by:

$$S_\chi(\mathcal{U}'_\Theta) = \{f : G'_A \to V_{\chi'}; f(agu) = \chi'(u_\infty)f(g)$$
$$\text{for any } g \in G'_A,\ u \in \mathcal{U}'_{\Theta'}, \text{ and } a \in G(Q)\ \}$$

where u_∞ is the component of u in G_∞. When $G'(R)$ modulo center is not compact, the definition of the cusp forms are as before.

Conjecture 1. For good choice of χ and χ' , the following relation between dimensions should hold:

$$\dim \Sigma_{\Theta \subsetneq S}(-1)^{\#(\Theta)} S_\chi(\mathcal{U}_\Theta) = \dim \Sigma_{\Theta' \subsetneq S'}(-1)^{\#(\Theta')} S_{\chi'}(\mathcal{U}_{\Theta'})$$

This is a natural problem to generalize classical results of Eichler on GL(2) and Shimizu on the product of GL(2).

Theorem(Hashimoto and Ibukiyama [2]) Put $G = GSp(2, Q)$ (size 4) and G' = the group of similitudes of positive definite binary quaternion hermitian forms on B^2 (B: the difinite quaternion algebra with prime discriminant p). Take as χ, or χ', the representation which corresponds to the young diagram parametrization (k,k), or (k-3,k-3), respectively. For

$v \neq p$, put $U_v = GSp(2, Z_p)$. Then for each prime $p \neq 3$, and each $k \geq 5$, Conjecture 1 is true.

We did not checked the case $p = 3$, just because calculation is complicated in that case. The results should be true also in this case. As for the weights less than 5, there exists a problem on the convergence of the trace formula, and the same argument in the above paper does not work.

To explain the meaning of the above equality, we need the difinition of new forms. We denote by $S^0_\chi(\mathcal{U}_\emptyset)$ the subspace of $S_\chi(\mathcal{U}_\emptyset)$ which is orthogonal (with respect to the usual invariant hermitian metric) to $\Sigma_{\Theta \subsetneqq S,\ \#(\Theta)=1} S_\chi(\mathcal{U}_\Theta)$. The space $S^0_{\chi'}(\mathcal{U}_\emptyset)$ is defined in the same way. We call elements of $S^0_\chi(\mathcal{U}_\emptyset)$ or $S^0_{\chi'}(\mathcal{U}'_\emptyset)$ new forms. In other words, f is a new form if and only if all the local representation π_v $(v \in V)$ attached to f is the Steinberg representation.

Conjecture 2. $S^0_\chi(\mathcal{U}_\emptyset) \cong S^0_{\chi'}(\mathcal{U}'_\emptyset)$ as $\mathcal{H}$-modules, where $\mathcal{H} = \otimes'_{v \notin V} \mathcal{H}(G_v, U_v)$ and $\mathcal{H}(G_v, U_v)$ are the usual (U_v-bi-invariant) Hecke algebras.

As for symplectic case, we have some numerical examples for rank 2 case. ([6]). For some groups of type A_2 of Q-rank 1, Koseki proved Conjecture 2 (, and hence also Conjecture 1) under some conditions on V and U_v ($v \notin V$).

Our method to approach to these problems is the trace formula. Since our approach is classical, this trace formula for G is the (generalization of) Godement's dimension formula in Cartan Seminar, and it is a summation of each conjugacy class of elements of $G(Q)$. Now, we assume that $G'(R)$ modulo center is compact. Then, any element of $G'(Q)$ is semi-simple, and taking the Langlands conjecture on stable conjugacy classes into account, any contribution of quasi-unipotent elements of $G(Q)$ to $\dim S^0_\chi(\mathcal{U}_\emptyset)$ should vanish. (We would like to emphasize that the contribution of quasi-unipotent elements to each $S_\chi(\mathcal{U}_\Theta)$ does not vanish in general. Only after taking the alternating sum, it should vanish.)

Conjecture 3: The contribution of central quasi-unipotent elements to $\dim S^0_\chi(\mathcal{U}_\emptyset)$ should vanish.

Theorem: As for central unipotent elements, this is true (for example) for G = Sp(n,Q) for general n. [5]

3 central quasi unipotent elements

From now on, we shall give some general program to solve Conjecture 3. The problem is on G and we forget about G'. From now on, we assume that G is a quasi-split semi-simple algebraic group over Q which is Q-simple and simply connected and that G(R) is associated with bounded symmetric domain. We denote by P_r (r = 1 ...s) the representatives of conjugacy classes of maximal Q-parabolic subgoups of G. We denote by U_r the Q-valued points of the center of the unipotent radical of P_r. As we assumed that $G(R)$ corresponds to bounded symmetric domain, we can assume that

$$U_1 \subset U_2 \subset \cdots \subset U_s$$

We say that an element γ of $G(Q)$ is quasi-unipotent, if some power of γ is unipotent. We say that quasi-unipotent element γ is central, if unipotent part γ_u of γ (in the Jordan decomposition) is conjugate to some elements of U_r for some r. Now, we define rank of quasi-unipotent elements $\gamma \in G(Q)$. For any such γ, we put

$$rank(\gamma) = min\{r; \text{some } G(Q)\text{-conjugate of } \gamma_u \text{ is in } U_r\}$$

Denote by $C^{qu,r}$ (resp.) $C^{u,r}$ the set of central quasi-unipotent (resp. unipotent) elements of $G(Q)$ of rank r. For any $\Theta \subset S$ and any subset C of Γ_Θ, we denote by $I(C,\Theta)$ the following integral:

$$I(C,\Theta) = \int_{\Gamma_\Theta \backslash G^0(R)} \sum_{\gamma \in C} \chi(J(g^{-1}\gamma g, o))dg$$

where o is the "origin" of the bounded domain D. This may be called the contribution of C to the dimension. Unfortunately, it is not known in general whether $I(C^{qu,r} \cap \Gamma_\Theta, \Theta)$ converges. The convergence of the Godement's formula (that is, the convergence of the integral expression $I(\Gamma_\Theta, \Theta)$) is easily obtained for generic χ. But, $I(C^{qu,r} \cap \Gamma_\Theta)$ is a part of whole integral and there is no a priori reason that this converges. Shintani proved that $I(C^{u,r} \cap \Gamma_\Theta)$ converges for $G = Sp(n,Q)$ (when χ is $\det^k$ with $k \geq 5$) by very subtle argument on prehomogeneous vector space, and also several other examples are known e.g. by Arakawa. Now, we assume that $I(C^{qu,r} \cap \Gamma_\Theta)$ converges.

Then, the problem becomes an arithmetic one. We treat this in the next section.

4 combinatorial theory

We fix $\Theta \subsetneq S$ for a while, and denote Γ_Θ simply by Γ. First, we decompose $C^{qu,r}$ into the part which corresponds with cusps. We decompose

G(Q) into the following finitely many double cosets:

$$G(Q) = \coprod_{w} \Gamma w P_r \quad \text{disjoint}$$

The cusps of Γ with respect to $P_r(Q)$ corresponds bijectively to the above double cosets. Further, for each representative w of cusps, we put

$$D^{qu,r}(w) = \{\gamma \in \Gamma \cap wP_r(Q)w^{-1} \cap C^{qu,r}; w^{-1}\gamma_u w \in U_r\}$$

and

$$C^{qu,r}(w) = \{\delta^{-1}\gamma\delta; \gamma \in D^{qu,r}(w), \delta \in \Gamma\}$$

Then, under a certain condition that any rank r unipotent element of U_r is, in a sense, "generic" in U_r (, which seems always true judging from various examples), we have the following decomposition:

$$C^{qu,r} \cap \Gamma = \coprod_{w} C^{qu,r}(w) \quad \text{disjoint}$$

where w runs over all the representatives of the cusps. So, it is enough to calculate $I(C^{qu,r}(w), \Theta)$ instead of $I(C^{qu,r} \cap \Gamma, \Theta)$ (of course under the assumption on convergence). Under the same condition as above, this integral depends only on $w^{-1}\Gamma w \cap P_r(Q)$. Hence, the conjecture is essentially reduces to the following problem.

Problem: We fix r and P_r. The set of pairs (Θ, w) (where $\Theta \subsetneqq S$ and w are the representatives of the cusps of Γ_Θ with respect to P_r) is decomposed into the disjoint union of the sets each of which consists of two elements (Θ_1, w_1) and (Θ_2, w_2) such that $w_1^{-1}\Gamma_{\Theta_1}w_1 \cap P_r(Q)$ is $P_r(Q)$-conjugate to $w_2^{-1}\Gamma_{\Theta_1}w_2 \cap P_r(Q)$ and that $\#(\Theta_1) = \#(\Theta_2) + 1$?

Now, we assume that $G_v(Q_v) = U_v P_r(Q_v)$ for any r and any $v \notin V$, and that $G_A = G(Q)\mathcal{U}_\Theta$ for all $\Theta \subsetneq S$. Then, the above problem reduces to the local problem. More precisely, fix $v \in V$. Put

$$\begin{aligned} T_\theta &= U_\theta \backslash G_v / P_r(Q_v) \text{ and} \\ T &= \coprod_{\theta \subsetneq S_{v,aff}} T_\theta \end{aligned}$$

Local Problem : Does there exist a permutation ι of T of order two such that the following two conditions (i) and (ii) are satisfied?

(i) If $c \in T_\theta$ and $\iota(c) \in T_\sigma$, then $\#(\theta) = \#(\sigma) + 1$.

(ii) Notations being as in (i), for any representative g (resp. h) in G_v of c (resp. $\iota(c)$), the group $g^{-1}U_\theta g \cap P_r(Q_v)$ is $P_r(Q_v)$-conjuugate to $h^{-1}U_\sigma h \cap P_r(Q_v)$.

This local problem can be solved affirmatively for various groups of type A_n or C_n, containing usual split symplectic groups. By the way, as a by-product to the solution of the above problem, we get a simultaneous description of explicit configurations of cusps of various Γ_{Θ} in the split symplectic case (cf. [5]).

As for more complete references, please see the references in the papers quoted below.

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